B ARITHMETIC SEQUENCES

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An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number. We call this number the **common difference** d.

A sequence is arithmetic $\Leftrightarrow$ $\boldsymbol{u_{n+1} - u_n = d}$ for all $n \in \mathbb{Z}^+$.

An arithmetic sequence can also be referred to as an **arithmetic progression**.

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An arithmetic sequence can also be referred to as an **arithmetic progression**.

For example:

- the tower of bricks in the previous Section forms an arithmetic sequence with common difference 1
- 2, 5, 8, 11, 14, ... is arithmetic with common difference 3 since
  $5 - 2 = 3$
  $8 - 5 = 3$
  $11 - 8 = 3$, and so on.
- 30, 25, 20, 15, 10, ... is arithmetic with common difference -5 since
  $25 - 30 = -5$
  $20 - 25 = -5$
  $15 - 20 = -5$, and so on.

The name "arithmetic" is given because the middle term of any three consecutive terms is the **arithmetic mean** of the terms on either side.

If the terms are a, b, c, then $b - a = c - b$ {equating common differences}

$$\therefore \ 2b = a + c$$

$$\therefore \ b = \frac{a+c}{2}$$ which is the arithmetic mean.

THE GENERAL TERM FORMULA

If we know that a sequence is arithmetic, we can use a formula to find the value of any term of the sequence.

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$, $u_3 = u_1 + 2d$, $u_4 = u_1 + 3d$, and so on.

Hence $$\underset{\text{term number}}{u_n} = u_1 + \underbrace{(n-1)}_{\substack{\text{the coefficient of } d \text{ is} \\ \text{one less than the term number}}} d$$

For an **arithmetic sequence** with **first term** $\boldsymbol{u_1}$ and **common difference** $\boldsymbol{d}$, the **general term** or $\boldsymbol{n}$**th term** is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 2 **Self Tutor**

Consider the sequence 2, 9, 16, 23, 30, ...

- **a** Show that the sequence is arithmetic.
- **b** Find a formula for the general term u_n.
- **c** Find the 100th term of the sequence.
- **d** Is **i** 828 **ii** 2341 a term of the sequence?

Example 2

Self Tutor

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find a formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a term of the sequence?

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a $9-2=7, \quad 16-9=7, \quad 23-16=7, \quad 30-23=7$

The difference between successive terms is constant.

$\therefore$ the sequence is arithmetic with $u_1 = 2$ and $d = 7$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 2 + 7(n-1) \\ \therefore\ u_n &= 7n - 5 \end{aligned}$$

c
$$\begin{aligned} u_{100} &= 7(100) - 5 \\ &= 695 \end{aligned}$$

d **i**
$$\begin{aligned} \text{Let}\quad u_n &= 828 \\ \therefore\ 7n - 5 &= 828 \\ \therefore\ 7n &= 833 \\ \therefore\ n &= 119 \end{aligned}$$

$\therefore$ 828 is a term of the sequence, and in fact is the 119th term.

ii
$$\begin{aligned} \text{Let}\quad u_n &= 2341 \\ \therefore\ 7n - 5 &= 2341 \\ \therefore\ 7n &= 2346 \\ \therefore\ n &= 335\tfrac{1}{7} \end{aligned}$$

But n must be an integer, so 2341 is not a member of the sequence.

1 For each of these arithmetic sequences:

- **i** State u_1 and d.
- **ii** Find the formula for the general term u_n.
- **iii** Find the 15th term of the sequence.

a 19, 25, 31, 37, ...

b 101, 97, 93, 89, ...

c 8, $9\frac{1}{2}$, 11, $12\frac{1}{2}$, ...

d 31, 36, 41, 46, ...

e 5, -3, -11, -19, ...

f a, $a+d$, $a+2d$, $a+3d$, ...

2 Consider the sequence 6, 17, 28, 39, 50, ...

a Show that the sequence is arithmetic.

b Find the formula for its general term.

c Find the 50th term.

d Is 325 a member?

e Is 761 a member?

3 Consider the sequence 87, 83, 79, 75, 71, ...

a Show that the sequence is arithmetic.

b Find the formula for its general term.

c Find the 40th term.

d Which term of the sequence is -297?

4 A sequence is defined by $u_n = 3n - 2$.

a By finding $u_{n+1} - u_n$, prove that the sequence is arithmetic.

b Find u_1 and d.

c Find the 57th term.

d What is the largest term of the sequence that is smaller than 450? Which term is this?

5 A sequence is defined by $u_n = \dfrac{71 - 7n}{2}$.

a Prove that the sequence is arithmetic.

b Find u_1 and d.

c Find u_{75}.

d For what values of n are the terms of the sequence less than -200?

6 An arithmetic sequence starts $23, 36, 49, 62, \ldots$. Find the first term of the sequence to exceed 100 000.

7 A sequence is defined by the formula $u_1 = -12, \quad u_{n+1} = u_n + 7, \quad n \geqslant 1$.

- **a** Prove that the sequence is arithmetic.
- **b** Find the 200th term of the sequence.
- **c** Is 1000 a member of the sequence?

Example 3 **Self Tutor**

Find k given that $3k+1$, k, and -3 are consecutive terms of an arithmetic sequence.

Example 3

Self Tutor

Find k given that $3k+1$, k, and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive, $k-(3k+1) = -3-k$ {equating differences}

$$\begin{aligned}
\therefore\quad k-3k-1 &= -3-k \\
\therefore\quad -2k-1 &= -3-k \\
\therefore\quad -1+3 &= -k+2k \\
\therefore\quad k &= 2
\end{aligned}$$

8 Find k given the consecutive arithmetic terms:

a $32,\ k,\ 3$ **b** $k,\ 7,\ 10$ **c** $k,\ 2k-1,\ 13$

d $k,\ 2k+1,\ 8-k$ **e** $2k+7,\ 3k+5,\ 5k-4$ **f** $2k+18,\ -2-k,\ 2k+2$

g $k,\ k^2,\ k^2+6$ **h** $5,\ k,\ k^2-8$

9 Suppose $10k+1$, $2k$, and $4k^2-5$ are consecutive terms of an arithmetic sequence.

a Find the possible values of k.

b For each value of k, find the common difference of the sequence.

Example 4

Self Tutor

Find the general term u_n for an arithmetic sequence with $u_3 = 8$ and $u_8 = -17$.

Example 4 **Self Tutor**

Find the general term u_n for an arithmetic sequence with $u_3 = 8$ and $u_8 = -17$.

$u_3 = 8 \quad \therefore \; u_1 + 2d = 8 \quad .... (1) \quad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_8 = -17 \quad \therefore \; u_1 + 7d = -17 \quad .... (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - 2d &= -8 \quad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 7d &= -17 \\ \hline \therefore \; 5d &= -25 \quad \{\text{adding the equations}\} \\ \therefore \; d &= -5 \end{aligned}$$

So, in (1):

$$\begin{aligned} u_1 + 2(-5) &= 8 \\ \therefore \; u_1 - 10 &= 8 \\ \therefore \; u_1 &= 18 \\ \text{Now} \quad u_n &= u_1 + (n-1)d \\ \therefore \; u_n &= 18 - 5(n-1) \\ \therefore \; u_n &= 18 - 5n + 5 \\ \therefore \; u_n &= 23 - 5n \end{aligned}$$

Check:

$$\begin{aligned} u_3 &= 23 - 5(3) \\ &= 23 - 15 \\ &= 8 \; \checkmark \\ u_8 &= 23 - 5(8) \\ &= 23 - 40 \\ &= -17 \; \checkmark \end{aligned}$$

10 Find the general term u_n for an arithmetic sequence with:

- **a** $u_7 = 41$ and $u_{13} = 77$
- **b** $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$
- **c** seventh term 1 and fifteenth term -39
- **d** eleventh and eighth terms being -16 and $-11\frac{1}{2}$ respectively.

11 Suppose a sequence $u_1, u_2, u_3, u_4, u_5, u_6, ...$ is arithmetic.

- **a** Show that $u_1 + u_2,\ u_3 + u_4,\ u_5 + u_6,\ ....$ is also arithmetic.
- **b** Given that $u_2 = 5$ and $u_{10} = 61$, find the 30th term of the sequence in **a**.

Example 5

Self Tutor

Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

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Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

Suppose the common difference is d.

$\therefore$ the numbers are $3,\ 3+d,\ 3+2d,\ 3+3d,\ 3+4d,$ and 12

$$\therefore\ 3+5d=12$$
$$\therefore\ 5d=9$$
$$\therefore\ d=\tfrac{9}{5}=1.8$$

So, the sequence is $3,\ 4.8,\ 6.6,\ 8.4,\ 10.2,\ 12.$

12 Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

13 Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

14
- **a** Insert three numbers between 50 and 44 so that all five numbers are in arithmetic sequence.
- **b** Assuming the sequence continues, find the first negative term of the sequence.

15 $\frac{1}{k}$, k, k^2+1 are respectively the 3rd, 4th, and 6th terms of an arithmetic sequence.

Given that $k \in \mathbb{Q}$, find: **a** k **b** the general term u_n.

16 The three numbers x, y, and z are such that $x > y > z > 0$.

Show that if $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ are consecutive terms of an arithmetic sequence, then $x - z$, y, and $x - y + z$ could be the lengths of the sides of a right angled triangle.

17 It is known that there are infinitely many prime numbers. Is it possible to construct an infinite arithmetic sequence such that all of the terms are distinct primes? Explain your answer.

Example 6

Self Tutor

Ryan is a cartoonist. His comic strip has just been bought by a newspaper, so he sends them the 28 comic strips he has drawn so far. Each week after the first he sends 3 more comic strips to the newspaper.

- **a** Find the total number of comic strips sent after 1, 2, 3, and 4 weeks.
- **b** Show that the total number of comic strips sent after n weeks forms an arithmetic sequence.
- **c** Find the number of comic strips sent after 15 weeks.
- **d** When does Ryan send his 120th comic strip?

Example 6

Self Tutor

Ryan is a cartoonist. His comic strip has just been bought by a newspaper, so he sends them the 28 comic strips he has drawn so far. Each week after the first he sends 3 more comic strips to the newspaper.

a Find the total number of comic strips sent after 1, 2, 3, and 4 weeks.

b Show that the total number of comic strips sent after n weeks forms an arithmetic sequence.

c Find the number of comic strips sent after 15 weeks.

d When does Ryan send his 120th comic strip?

a *Week 1*: 28 comic strips
Week 2: $28 + 3 = 31$ comic strips
Week 3: $31 + 3 = 34$ comic strips
Week 4: $34 + 3 = 37$ comic strips

b Every week, Ryan sends 3 comic strips, so the difference between successive weeks is always 3. We have an arithmetic sequence with $u_1 = 28$ and common difference $d = 3$.

c
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 28 + (n-1) \times 3 \\ &= 25 + 3n \end{aligned} \qquad \therefore \quad \begin{aligned} u_{15} &= 25 + 3 \times 15 \\ &= 70 \end{aligned}$$

After 15 weeks Ryan has sent 70 comic strips.

d We want to find n such that
$$\begin{aligned} u_n &= 120 \\ \therefore \quad 25 + 3n &= 120 \\ \therefore \quad 3n &= 95 \\ \therefore \quad n &= 31\tfrac{2}{3} \end{aligned}$$

Ryan sends the 120th comic strip in the 32nd week.

18 A luxury car manufacturer sets up a factory for a new model vehicle. In the first month only 5 cars are made. After this, 13 cars are made every month.

- **a** List the total number of cars that have been made by the end of each of the first six months.
- **b** Explain why the total number of cars made after n months forms an arithmetic sequence.
- **c** How many cars are made in the first year?
- **d** How long is it until the 250th car is made?

19 At the start of the dry season, Yafiah's 3000 L water tank is full. She uses 183 L of water each week to water her garden.

- **a** Find the amount of water left in the tank after 1, 2, 3, and 4 weeks.
- **b** Explain why the amount of water left in the tank after n weeks forms an arithmetic sequence.
- **c** When will Yafiah's tank run out of water?

1 **a** **i** $u_1 = 19,\ d = 6$ **ii** $u_n = 6n + 13$ **iii** $u_{15} = 103$

b **i** $u_1 = 101,\ d = -4$ **ii** $u_n = 105 - 4n$
iii $u_{15} = 45$

c **i** $u_1 = 8,\ d = 1\frac{1}{2}$ **ii** $u_n = 1\frac{1}{2}n + 6\frac{1}{2}$ **iii** $u_{15} = 29$

d **i** $u_1 = 31,\ d = 5$ **ii** $u_n = 5n + 26$ **iii** $u_{15} = 101$

e **i** $u_1 = 5,\ d = -8$ **ii** $u_n = 13 - 8n$
iii $u_{15} = -107$

f **i** $u_1 = a,\ d = d$ **ii** $u_n = a + (n-1)d$
iii $u_{15} = a + 14d$

2 **a** $u_1 = 6,\ d = 11$ **b** $u_n = 11n - 5$
c $u_{50} = 545$ **d** yes, $u_{30} = 325$ **e** no

3 **a** $u_1 = 87,\ d = -4$ **b** $u_n = 91 - 4n$
c $u_{40} = -69$ **d** u_{97}

4 **b** $u_1 = 1,\ d = 3$ **c** $u_{57} = 169$ **d** $u_{150} = 448$

5 **b** $u_1 = 32,\ d = -\frac{7}{2}$ **c** $u_{75} = -227$ **d** $n \geqslant 68$

6 $u_{7692} = 100\,006$ **7** **b** $u_{200} = 1381$ **c** no

8 **a** $k = 17\frac{1}{2}$ **b** $k = 4$ **c** $k = 5$ **d** $k = \frac{3}{2}$
e $k = 7$ **f** $k = -4$ **g** $k = -2$ or 3 **h** $k = -1$ or 3

9 **a** $k = \frac{1}{2}$ or -2 **b** For $k = \frac{1}{2},\ d = -5$
For $k = -2,\ d = 15$

10 **a** $u_n = 6n - 1$ **b** $u_n = -\frac{3}{2}n + \frac{11}{2}$
c $u_n = -5n + 36$ **d** $u_n = -\frac{3}{2}n + \frac{1}{2}$

11 **b** $u_{30} = 815$ **12** $5,\ 6\frac{1}{4},\ 7\frac{1}{2},\ 8\frac{3}{4},\ 10$

13 $-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32$

14 **a** $50,\ 48\frac{1}{2},\ 47,\ 45\frac{1}{2},\ 44$ **b** $u_{35} = -1$

15 **a** $k = 2$ **b** $u_n = \dfrac{3n - 8}{2}$

16 **Hint:** Since $x > y > z > 0$, then $\frac{1}{x} < \frac{1}{y} < \frac{1}{z}$.

17 No, such a sequence would require $u_1 =$ some prime p, and $d \in \mathbb{Z}^+$. But then u_{p+1} must be a multiple of p.

18 **a** Month 1: 5 cars Month 4: 44 cars
Month 2: 18 cars Month 5: 57 cars
Month 3: 31 cars Month 6: 70 cars

b The total number of cars made increases by 13 each month. So, the common difference $d = 13$.

c 148 cars **d** 20 months

19 **a** Week 1: 2817 L Week 3: 2451 L
Week 2: 2634 L Week 4: 2268 L

b The amount in the tank decreases by the same amount (183 L) each week.

c in the 17th week